Orthomodular Lattices and Beyond

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An *orthولattice (OL)* is an algebra $\langle A, \cup, \cap, ' >$ in which the following conditions hold:

$$
\begin{aligned}
a \cup b &= b \cup a && (1)\\
(a \cup b) \cup c &= a \cup (b \cup c) && (2)\\
a'' &= a && (3)\\
a \cup (a \cap b) &= a && (4)\\
a \cap b &= (a' \cup b')' && (5)
\end{aligned}
$$

An *orthomodular lattice (OML)* is an OL in which

$$
a \cup b = ((a \cup b) \cap b') \cup b \qquad (6)
$$

A *Boolean algebra (BA)* is an OML in which

$$
a = (a \cap b) \cup (a \cap b') \qquad (7)
$$

A Neat Result

There is a single axiom of length 23 for OML, where $a|b \stackrel{\text{def}}{=} a' \cup b'$ is the Sheffer stroke (McCune, Rose, Veroff http://www.mcs.anl.gov/~mccune/papers/olsax/):

$$((((b|a)|(a|c))|d)|(a|((c|((a|a)|c))|c))) \;=\; a \tag{8}$$

Open problem(?): is there one of length 21?

Some Definitions

$$1 \overset{\text{def}}{=} a \cup a' \quad \text{(unit)} \tag{9}$$

$$0 \overset{\text{def}}{=} 1' \quad \text{(zero)} \tag{10}$$

$$a \le b \overset{\text{def}}{\Leftrightarrow} a = a \cap b \quad \text{(less-than-or-equal)} \tag{11}$$

$$a \equiv b \overset{\text{def}}{=} (a \cup b) \cap (a' \cup b') \quad \text{(equivalence)} \tag{12}$$

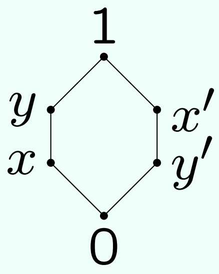
$$a \,\mathsf{C}\, b \overset{\text{def}}{\Leftrightarrow} a = (a \cap b) \cup (a \cap b') \quad \text{(commutes)} \tag{13}$$

A *weakly orthomodular lattice* (WOML) is an OL in which

$$(a' \cap (a \cup b)) \cup b' \cup (a \cap b) \ = \ 1 \tag{14}$$

A *weakly Boolean algebra* (WBA) is a WOML in which

$$a' \cup (a \cap b) \cup (a \cap b') \ = \ 1 \tag{15}$$



Lattice O6 is an example of a WOML and a WBA. It is non-orthomodular and non-distributive, yet it is a model for both quantum and classical propositional calculus!

Summary of WOML and WBA Results

- All OMLs (BAs) are WOMLs (WBAs).

- Not all WOMLs (WBAs) are OMLs (BAs).

- Any OML (BA) equation can be represented in WOML (WBA) with the following mapping:

OML (BA)	WOML (WBA)
$a = b$	$a \equiv b = 1$

- WOMLs (WBAs) are more general models for quantum (classical) propositional calculus, than the usual OML (BA) models.

The 6 Implications in OMLs

$$a \to_0 b \stackrel{\text{def}}{=} a' \cup b \quad \text{(classical)} \tag{16}$$

$$a \to_1 b \stackrel{\text{def}}{=} a' \cup (a \cap b) \quad \text{(Sasaki)} \tag{17}$$

$$a \to_2 b \stackrel{\text{def}}{=} b' \to_1 a' \quad \text{(Dishkant)} \tag{18}$$

$$a \to_3 b \stackrel{\text{def}}{=} (a' \cap b) \cup (a' \cap b') \cup (a \to_1 b) \quad \text{(Kalmbach)} \tag{19}$$

$$a \to_4 b \stackrel{\text{def}}{=} b' \to_3 a' \quad \text{(non-tollens)} \tag{20}$$

$$a \to_5 b \stackrel{\text{def}}{=} (a \cap b) \cup (a' \cap b) \cup (a' \cap b') \quad \text{(relevance)} \tag{21}$$

All 6 implications evaluate to $a \to_0 b$ in a Boolean lattice. $\to_i$, for $i \neq 0$, is called a *quantum* implication. $\to_0$ is called a *classical* implication.

Quantum implications are distinguished by the fact in an OML they satisfy the *Birkhoff-von Neumann condition*:

$$a \rightarrow_i b = 1 \quad \Leftrightarrow \quad a \leq b, \qquad i = 1, \ldots, 5 \tag{22}$$

Neat result (Pavičić/Megill, 1998):

$$a \cup b \;=\; (a \rightarrow_i b) \rightarrow_i (((a \rightarrow_i b) \rightarrow_i (b \rightarrow_i a)) \rightarrow_i a) \tag{23}$$

holds in any OML for $i = 1, \ldots, 5$. This observation lets us to construct, by adding a constant 0, an OML-equivalent "unified" algebra with an (unspecified) quantum implication as its only binary operation. Thus, we can study the properties common to all quantum implications without a philosophical debate of which is the "real" implication.

Orthoimplication Algebra $\langle A, . \rangle$ (Abbott, 1976)

$$(ab)a = a \tag{24}$$

$$(ab)b = (ba)a \tag{25}$$

$$a((ba)c) = ac \tag{26}$$

If "." is interpreted as $\to_2$, then each equation holds in OML.

Conjecture (completeness): All such equations (i.e. polynomials in $\to_2$ on each side of equality) that hold in OML can be proved from this algebra.

Quasi-Implication Algebra $\langle A, .\rangle$ (Hardegree, 1981)

$$(ab)a = a \tag{27}$$

$$(ab)(ac) = (ba)(bc) \tag{28}$$

$$((ab)(ba))a = ((ba)(ab))b \tag{29}$$

If "." is interpreted as $\to_1$, then each equation holds in OML.

Theorem (completeness) [Hardegree]: All such equations that hold in OML, with $\to_1$ as the only operation, can be proved from this algebra.

Other Implication Algebras

Similar algebras for $\rightarrow_3$, $\rightarrow_4$, and $\rightarrow_5$ have not been proposed nor any completeness results obtained.

The most promising system for future study is $\rightarrow_5$, because $a \rightarrow_1 b = a \rightarrow_5 (a \rightarrow_5 b)$ in any OML, holding promise that the ideas in Hardegree's $\rightarrow_1$ proof can be adapted for $\rightarrow_5$.

Systems for $\rightarrow_3$ and $\rightarrow_4$, as well as completeness of Abbott's $\rightarrow_2$ system, remain complete mysteries.

Another Open OML Problem

Problem: does the following equation hold in all OMLs?

$$(a \to_5 b) \cap (b \to_5 c) \cap (c \to_5 d) \cap (d \to_5 e) \cap (e \to_5 a) = (a \equiv b) \cap (b \equiv c) \cap (c \equiv d) \cap (d \equiv e) \quad (30)$$

Note: It holds for ≤ 4 variables. It does not hold for ≥ 6 variables.

Orthomodular Lattices and Hilbert Space

Fact: The OML axioms hold in the lattice of closed subspaces of infinite dimensional Hilbert space, $\mathcal{C}(\mathcal{H})$. This is a primary motivation for studying them. But they aren't the only equations that hold!

Some history:

- 1936 - Birkhoff/von Neumann attempt to find a "logical structure" for quantum mechanics, but find only the modular law (holding only for finite-dimensional Hilbert space).

- 1937 - Husumi discovers the orthomodular law and shows that it holds in $\mathcal{C}(\mathcal{H})$.

History (cont.)

- 1975 - Day discovers the orthoarguesian law and shows that it holds in $\mathcal{C}(\mathcal{H})$.

- 1981 - Godowski discovers an infinite equational variety derived from properties of states on $\mathcal{C}(\mathcal{H})$, that holds in $\mathcal{C}(\mathcal{H})$.

- 1985 - Mayet extends Godowski's discovery to prove the existence of a more general equational variety that holds in $\mathcal{C}(\mathcal{H})$. (However Mayet provides no actual examples of these new equations that are stronger than Godowski's.)

History (cont.)

- 1995 - Solèr proves that an OML, with certain additional conditions, determines a Hilbert space (very significant). Thus OML theory (with these conditions) and Hilbert space theory are duals.

Some recent results:

- 2000 - Megill/Pavičić found an infinite equational variety related to orthoarguesian equations, but stronger, that holds in $\mathcal{C}(\mathcal{H})$.

- 2003 - Megill/Pavičić found examples (unpublished) of Mayet's equations that are stronger than Godowski's, that hold in $\mathcal{C}(\mathcal{H})$.

Equations Related to States That Hold in $\mathcal{C}(\mathcal{H})$

The simplest *Godowski equation* is

$$(a \to_1 b) \cap (b \to_1 c) \cap (c \to_1 a) \ \leq \ a \to_1 c \tag{31}$$

Using Mayet's theory, Megill/Pavičić (unpublished) found examples of equations stronger than (independent from) Godowski's. The simplest example is

$$((a \to_1 b) \to_1 (c \to_1 b)) \cap (a \to_1 c) \cap (b \to_1 a) \ \leq \ c \to_1 a \tag{32}$$

More Definitions

$$a \stackrel{c}{\equiv} b \stackrel{\text{def}}{=} ((a \to_1 c) \cap (b \to_1 c)) \cup ((a' \to_1 c) \cap (b' \to_1 c)) \quad (33)$$

$$a \stackrel{c,d}{\equiv} b \stackrel{\text{def}}{=} (a \stackrel{d}{\equiv} b) \cup ((a \stackrel{d}{\equiv} c) \cap (b \stackrel{d}{\equiv} c)) \quad (34)$$

Orthoarguesian Equations That Hold in $\mathcal{C}(\mathcal{H})$

$$(a \to_1 c) \cap (a \stackrel{c}{\equiv} b) \leq b \to_1 c \quad \text{(OA3)} \quad (35)$$

$$(a \to_1 d) \cap (a \stackrel{c,d}{\equiv} b) \leq b \to_1 d \quad \text{(OA4)} \quad (36)$$

OA4 is a 4-variable equivalent to Day's original 6-variable orthoarguesian equation. OA3 is a strictly weaker 3-variable equation, that is still stronger than the OM law. OMLs in which OA3 or OA4 hold are called 3OAs, 4OAs respectively.

Generalization of Orthoarguesian Law (Definitions)

$$a_1 \overset{(3)}{\equiv} a_2 \overset{\text{def}}{=} a_1 \overset{a_3}{\equiv} a_2 \tag{37}$$

$$a_1 \overset{(4)}{\equiv} a_2 \overset{\text{def}}{=} a_1 \overset{a_4,a_3}{\equiv} a_2 \tag{38}$$

$$a_1 \overset{(5)}{\equiv} a_2 \overset{\text{def}}{=} (a_1 \overset{(4)}{\equiv} a_2) \cup ((a_1 \overset{(4)}{\equiv} a_5) \cap (a_2 \overset{(4)}{\equiv} a_5)) \tag{39}$$

$$a_1 \overset{(n)}{\equiv} a_2 \overset{\text{def}}{=} (a_1 \overset{(n-1)}{\equiv} a_2) \cup ((a_1 \overset{(n-1)}{\equiv} a_n) \cap (a_2 \overset{(n-1)}{\equiv} a_n)), \quad n \geq 4 \tag{40}$$

Generalization of Orthoarguesian Law (Definitions, cont.)

To obtain $\stackrel{(n)}{\equiv}$ we substitute in each $\stackrel{(n-1)}{\equiv}$ subexpression only the two explicit variables, leaving the other variables the same.

For example, $(a_2 \stackrel{(4)}{\equiv} a_5)$ in (39) means

$(a_2 \stackrel{(3)}{\equiv} a_5) \cup ((a_2 \stackrel{(3)}{\equiv} a_4) \cap (a_5 \stackrel{(3)}{\equiv} a_4))$ which means

$(((a_2 \to_1 a_3) \cap (a_5 \to_1 a_3)) \cup ((a_2' \to_1 a_3) \cap (a_5' \to_1 a_3))) \cup ((((a_2 \to_1 a_3) \cap (a_4 \to_1 a_3)) \cup ((a_2' \to_1 a_3) \cap (a_4' \to_1 a_3))) \cap (((a_5 \to_1 a_3) \cap (a_4 \to_1 a_3)) \cup ((a_5' \to_1 a_3) \cap (a_4' \to_1 a_3))))$

Generalization of Orthoarguesian Law (cont.)

Theorem [Megill/Pavičić, 2000]: The nOA *laws*

$$(a_1 \to_1 a_3) \cap (a_1 \overset{(n)}{\equiv} a_2) \le a_2 \to_1 a_3 \,. \tag{41}$$

hold in $\mathcal{C}(\mathcal{H})$. In addition, they form a series of successively stronger laws than 3OA and 4OA (proved for $n = 5$ and $n = 6$; open problem for $n > 6$).

The independence proof for $n = 6$ required 10 CPU years on a 192-CPU Linux cluster at Australian National University.

“Orthoarguesian Identity” Laws

The relations

$$a \overset{c}{\equiv} b = 1 \qquad \Leftrightarrow \qquad a \to_1 c = b \to_1 c \qquad \text{(OI3)} \qquad (42)$$

$$a \overset{c,d}{\equiv} b = 1 \qquad \Leftrightarrow \qquad a \to_1 d = b \to_1 d \qquad \text{(OI4)} \qquad (43)$$

hold in all 3OAs, 4OAs respectively.

Open problems:

OI3 *conjecture*: All OMLs in which OI3 holds are 3OAs.

OI4 *conjecture*: All OMLs in which OI4 holds are 4OAs.

$100 Prize

I have “wasted” so much time and effort over the past 3 years trying to prove or disprove the OI3 conjecture that, in an effort to maintain my sanity, I hereby offer this prize to anyone who proves or disproves it.

The following equation, if it holds in all OMLs, will prove the OI3 conjecture (note that $-a$ means a'):

$$((((-(-a \cup (a \cap c)) \cup ((-(-a \cup (a \cap c)) \cup -(-b \cup (b \cap c))) \cap (-a \cup -b))) \cup (((-a \cup (a \cap c)) \cap (((-a \cup (a \cap c)) \cap (-b \cup (b \cap c))) \cup (a \cap b))) \cap c)) \cap ((-(-b \cup (b \cap c)) \cup ((-(-a \cup (a \cap c)) \cup -(-b \cup (b \cap c))) \cap (-a \cup -b))) \cup (((-b \cup (b \cap c)) \cap (((-a \cup (a \cap c)) \cap (-b \cup (b \cap c))) \cup (a \cap b))) \cap c))) \cup ((-a \cup (a \cap c)) \cap (-b \cup (b \cap c)))) = 1$$

Using the Sasaki implication, we can abreviate this as follows:

$$(((((a \to_1 c) \cap (((a \to_1 c) \cap (b \to_1 c)) \cup (a \cap b))) \to_1 c) \cap (((b \to_1 c) \cap (((a \to_1 c) \cap (b \to_1 c)) \cup (a \cap b))) \to_1 c)) \cup ((a \to_1 c) \cap (b \to_1 c))) = 1$$

Equations Related to the OI3 Conjecture

$$
\begin{align}
(a \to_1 c) \cap (a \overset{c}{\equiv} b) \quad &\mathsf{C} \quad b \to_1 c \tag{44}\\
(a \to_1 c) \cap (a \overset{c}{\equiv} b) \quad &\mathsf{C} \quad (b \to_1 c) \cap (a \overset{c}{\equiv} b) \tag{45}\\
(a' \to_1 c)' \cap (a \overset{c}{\equiv} b) \quad &\le \quad b \to_1 c \tag{46}\\
(a' \to_1 c)' \cap (a \overset{c}{\equiv} b) \quad &\mathsf{C} \quad b \to_1 c \tag{47}\\
(a' \to_1 c)' \cap (a \overset{c}{\equiv} b) \quad &\mathsf{C} \quad (b \to_1 c) \cap (a \overset{c}{\equiv} b) \tag{48}\\
c \cap (a \to_1 c) \cap (a \overset{c}{\equiv} b) \quad &\le \quad (b \to_1 c) \tag{49}\\
c \cap (a \to_1 c) \cap (a \overset{c}{\equiv} b) \quad &\mathsf{C} \quad (b \to_1 c) \tag{50}\\
c \cap (a \to_1 c) \cap (a \overset{c}{\equiv} b) \quad &\mathsf{C} \quad (b \to_1 c) \cap (a \overset{c}{\equiv} b) \tag{51}\\
((a \to_1 c) \cap (a \overset{c}{\equiv} b)) \to_1 c \quad &= \quad ((b \to_1 c) \cap (a \overset{c}{\equiv} b)) \to_1 c \tag{52}\\
((a \to_1 c) \cap (a \overset{c}{\equiv} b)) \to_1 c \quad &\mathsf{C} \quad ((b \to_1 c) \cap (a \overset{c}{\equiv} b)) \to_1 c \tag{53}
\end{align}
$$

Equations Related to the OI3 Conjecture (cont.)

All of these equations are implied by the 3OA law. All of these equations imply OI3. Unknown is whether most of them are equivalent to the 3OA law. A proof of the OI3 conjecture would establish all of them as equivalent to the 3OA law. Known results are as follows (note that $\Rightarrow$ means "can be proved from the axiom system of OML + the left-hand side equation added as an axiom"):

$$\text{OA3} \Leftrightarrow 44 \Rightarrow 45 \Rightarrow \text{OI3}$$
$$\text{OA3} \Rightarrow 46 \Rightarrow 47 \Rightarrow 48 \Rightarrow \text{OI3}$$
$$\text{OA3} \Rightarrow 49 \Leftrightarrow 50 \Leftrightarrow 51 \Rightarrow \text{OI3}$$
$$\text{OA3} \Leftrightarrow 52 \Rightarrow 53 \Rightarrow \text{OI3}$$

References

Most of the references for this material can be found at:

`http://us.metamath.org/qlegif/mmql.html#ref`

More miscellaneous stuff can be found at:

`http://users.shore.net/~ndm/award2003.html`